

Variants into minimal logic of the Kuroda negative translation*

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Abstract

We present three variants of the Kuroda negative translation that translate classical logic into minimal logic. Also, we discuss how they affect the methods of extracting computational content from proofs.

1 Outline

The Kuroda negative translation K [20, page 46] maps each formula A to the formula $A^K := \neg\neg A_K$, where A_K is obtained from A by adding a double negation after each universal quantification. The translation K translates classical logic into intuitionistic logic, but not into minimal logic. We present three variants K_1 , K_2 and K_3 of K that do translate into minimal logic. They differ from K in the following (where $P \neq \perp$ is an atomic formula):

- $P_{K_1} := P \vee \perp$;
- $P_{K_2} := \neg\neg P$;
- $(A \rightarrow B)_{K_3} := \neg A_{K_3} \vee B_{K_3}$.

We discuss how these variants affect two methods of extracting computational content from proofs:

- (functional interpretation) \circ (negative translation);
- (realisability) \circ (Friedman A -translation) \circ (negative translation).

We conclude that each method performs equally well with K , K_1 and K_2 , but differently (sometimes better, sometimes worse) with K_3 .

*Keywords: negative translation, classical logic, intuitionistic logic, minimal logic, functional interpretation, realisability, Friedman A -translation.

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2 Background

2.1 Classical, intuitionistic and minimal logics

Informally speaking, classical, intuitionistic and minimal logics are the following.

- Classical logic **CL** is the usual logic in mathematics.
- Intuitionistic logic **IL** is **CL** without:

- proof by contradiction $\frac{\neg A}{\perp}$;
- law of excluded middle $A \vee \neg A$;
- law of double negation $\neg\neg A \rightarrow A$.

- Minimal logic **ML** is **IL** without ex falso quodlibet $\perp \rightarrow A$.

We base the language of **CL**, **IL** and **ML** on \perp , \wedge , \vee , \rightarrow , \forall and \exists , and we define $\neg A \equiv A \rightarrow \perp$ and $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$. We make the convention that \forall , \exists and \neg bind stronger than \wedge and \vee , which in turn bind stronger than \rightarrow and \leftrightarrow .

2.2 Negative translations

We can embed **CL** into **IL** by the so-called negative translations. A *negative translation* N is a function, mapping a formula A to a formula A^N , that faithfully embeds **CL** into **IL** in the sense of holding the following soundness and characterisation.

Soundness For all formulas A , we have: if $\mathbf{CL} + \Gamma \vdash A$, then $\mathbf{IL} + \Gamma^N \vdash A^N$ (where $\Gamma^N := \{A^N : A \in \Gamma\}$).

Characterisation For all formulas A , we have $\mathbf{CL} \vdash A \leftrightarrow A^N$.

There are four negative translations usually found in the literature:

- the Kolmogorov negative translation [19] [18, formula (49)];
- the Gödel-Gentzen negative translation [12] [14, page 287] [9] [10, theorem III];
- the Kuroda negative translation [20, page 46];
- the Krivine negative translation [24, sections 2 and 4] [1, page 1].

All, except the Kuroda negative translation, translate **CL** even into **ML** (that is the soundness holds if we replace in it **IL** by **ML**). For the convenience of the reader, we recall the definition of the Kuroda negative translation.

Definition 1. The *Kuroda negative translation* K maps each formula A to the formula $A^K \equiv \neg\neg A_K$, where A_K is defined by induction on the structure of formulas by

$$\begin{aligned} P_K &\equiv P, & (A \wedge B)_K &\equiv A_K \wedge B_K, \\ (\forall x A)_K &\equiv \forall x \neg\neg A_K, & (A \vee B)_K &\equiv A_K \vee B_K, \\ (\exists x A)_K &\equiv \exists x A_K, & (A \rightarrow B)_K &\equiv A_K \rightarrow B_K, \end{aligned}$$

where P is an atomic formula.

2.3 Extracting computational content

Negative translations are used in the process of extracting computational content from proofs. There are two competing methods of extracting computational content. Schematically, these methods are the following compositions of translations:

functional interpretation method = (functional interpretation) \circ
(negative translation),

realisability method = (realisability) \circ
(Friedman A -translation) \circ
(negative translation).

For concreteness, we take the functional interpretation as being the Gödel functional interpretation D [13] [15, page 249] (there is a modern treatment of it in Kohlenbach's book [17, chapter 8]).

There are two variants of the Friedman A -translation. Let us fix a formula A .

- The original *Friedman A -translation* F [8, section 1] [5] [4, page 463] maps each formula B to the formula B^F obtained from B by simultaneously substituting in B :
 - the atomic subformulas $P \neq \perp$ by $P \vee A$;
 - \perp by A .

This translation is sound in the sense of: if $\text{IL} \vdash B$, then $\text{ML} \vdash B^F$.

- The *refined Friedman A -translation* rF [3, lemma 2.1] maps each formula B to the formula B^{rF} obtained from B by simultaneously substituting in B all occurrences of \perp by A .

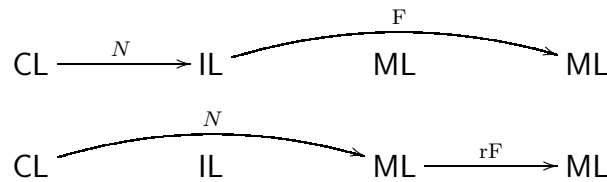
This translation is sound in the sense of: if $\text{ML} \vdash B$, then $\text{ML} \vdash B^{\text{rF}}$.

Let us remark the differences between the two translations:

- F makes more complicated substitutions, but translates even IL into ML;
- rF makes simpler substitutions, but translates only ML into ML.

In the realisability method we can use F or rF. The criteria to choose between the two is the following:

- we use F when the negative translation N translates CL only into IL;
- we use rF when the negative translation N translates CL even into ML.



Accordingly, if N is K then we use F, and if N is a variant into ML of K then we use rF.

3 First variant of the Kuroda negative translation

3.1 Motivation

The simplest instance of F is the case $A \equiv \perp$, since in this case the substitution of \perp by A is unnecessary. So, for $A \equiv \perp$, the translation F reduces to the translation T_1 , of IL into ML , that simply substitutes atomic subformulas $P \not\equiv \perp$ by $P \vee \perp$. If we compose K with T_1 , then we get a translation $K_1 = T_1 \circ K$ from CL into ML .

$$\begin{array}{ccccc} CL & \xrightarrow{K} & IL & \xrightarrow{T_1} & ML \\ & \searrow & & \nearrow & \\ & & K_1 & & \end{array}$$

3.2 Translation T_1

Definition 2 ([21, page 686] [22, section 6.2]). The translation T_1 maps each formula A to the formula A^{T_1} obtained from A by simultaneously substituting in A all atomic subformulas $P \not\equiv \perp$ by $P \vee \perp$.

Theorem 3 (soundness of T_1 [21, page 686] [22, section 6.2]). *For all formulas A , we have: if $IL \vdash A$, then $ML \vdash A^{T_1}$.*

Proof. The translation T_1 is F with $A \equiv \perp$. The soundness theorem is known for the F (with some conditions on free and bounded variables that are satisfied by $A \equiv \perp$).

Alternatively, we do the proof by induction on derivations. The translation by T_1 of an axiom or a rule of IL (as axiomatised in Kohlenbach's book [17, section 3.1]) is an instance of that axiom or rule. So it only remains to see that ML proves $(\perp \rightarrow A)^{T_1}$, that is $\perp \rightarrow A^{T_1}$. This is done noting that A^{T_1} is built from “atoms” \perp and $P \vee \perp$ by means of the connectives $\wedge, \vee, \rightarrow, \forall$ and \exists ; and that \perp implies in ML the “atoms” and these implications are preserved by the connectives. \square

Theorem 4 (characterisation of T_1). *For all formulas A , we have $IL \vdash A \leftrightarrow A^{T_1}$.*

Proof. Just note that A^{T_1} is obtained from A by substituting in A the atomic subformulas $P \not\equiv \perp$ by the equivalent (in IL) $P \vee \perp$. \square

3.3 Translation K_1

Definition 5. The *first variant* K_1 of the Kuroda negative translation maps each formula A to the formula $A^{K_1} := \neg\neg A_{K_1}$, where A_{K_1} is defined by induction on the structure of formulas by

$$\begin{aligned} P_{K_1} &:= P \vee \perp, & (A \wedge B)_{K_1} &:= A_{K_1} \wedge B_{K_1}, \\ \perp_{K_1} &:= \perp, & (A \vee B)_{K_1} &:= A_{K_1} \vee B_{K_1}, \\ (\forall x A)_{K_1} &:= \forall x \neg\neg A_{K_1}, & (A \rightarrow B)_{K_1} &:= A_{K_1} \rightarrow B_{K_1}, \\ (\exists x A)_{K_1} &:= \exists x A_{K_1}, \end{aligned}$$

where $P \not\equiv \perp$ is an atomic formula.

Remark 6. The only difference between K and K_1 is in the clause of P .

Proposition 7 (factorisation $K_1 = T_1 \circ K$). *For all formulas A , we have $A^{K_1} \equiv (A^K)^{T_1}$.*

Proof.

1. First, we prove $(A^K)^{T_1} \equiv A_{K_1}$ by induction on the structure of formulas. Let us consider only the cases of atomic formulas $P \not\equiv \perp, \wedge$ and \forall . The remaining cases are analogous. In the following, “IH” marks a use of induction hypothesis. It is convenient to note $(\neg A)^{T_1} \equiv \neg A^{T_1}$. We have:

$$\begin{array}{lll}
(P_K)^{T_1} \equiv & ((A \wedge B)_K)^{T_1} \equiv & ((\forall x A)_K)^{T_1} \equiv \\
P^{T_1} \equiv & (A_K \wedge B_K)^{T_1} \equiv & (\forall x \neg \neg A_K)^{T_1} \equiv \\
P \vee \perp \equiv & (A_K)^{T_1} \wedge (B_K)^{T_1} \stackrel{\text{IH}}{\equiv} & \forall x \neg \neg (A_K)^{T_1} \stackrel{\text{IH}}{\equiv} \\
P_{K_1}, & A_{K_1} \wedge B_{K_1} \equiv & \forall x \neg \neg A_{K_1} \equiv \\
& (A \wedge B)_{K_1}, & (\forall x \neg \neg A)_{K_1}.
\end{array}$$

2. Now we prove $(A^K)^{T_1} \equiv A^{K_1}$ using $(A_K)^{T_1} \equiv A_{K_1}$:

$$(A^K)^{T_1} \equiv (\neg \neg A_K)^{T_1} \equiv \neg \neg (A_K)^{T_1} \equiv \neg \neg A_{K_1} \equiv A^{K_1}. \quad \square$$

Theorem 8 (soundness of K_1). *For all formulas A , we have: if $\text{CL} \vdash A$, then $\text{ML} \vdash A^{K_1}$.*

Proof. Having in mind the factorisation $K_1 = T_1 \circ K$, just compose the soundness theorems of K and T_1 . \square

Theorem 9 (characterisation of K_1). *For all formulas A , we have $\text{CL} \vdash A \leftrightarrow A^{K_1}$.*

Proof. Having in mind the factorisation $K_1 = T_1 \circ K$, just compose the characterisation theorems of T_1 and K . \square

Theorem 10 (equivalence between K and K_1).

1. *For all formulas A , we have $\text{IL} \vdash A^K \leftrightarrow A^{K_1}$.*
2. *The previous point does not hold in ML.*

Proof.

1. Having in mind the factorisation $K_1 = T_1 \circ K$, just use the characterisation theorem of T_1 .
2. Recall that K does not translate CL into ML but K_1 does so, thus K and K_1 cannot be equivalent in ML . \square

3.4 Effect on extracting computational content

Let us recall that the only difference between K and K_1 is $P_K \equiv P$ and $P_{K_1} \equiv P \vee \perp$ (where $P \not\equiv \perp$ is an atomic formula).

Functional interpretation method To compare the functional interpretation method with K versus K_1 , we only have to compare P^D versus $(P \vee \perp)^D$:

- $P^D \equiv P$;
- $(P \vee \perp)^D \equiv \exists n((n = 0 \rightarrow P) \wedge (n \neq 0 \rightarrow \perp))$, where we can witness n by 0, and doing so $(P \vee \perp)^D$ becomes equivalent to P .

We conclude that the functional interpretation method with K versus K_1 is (essentially) the same.

Realisability method To compare the realisability method with K versus K_1 , we only have to compare P^F versus $(P \vee \perp)^{rF}$:

- $P^F \equiv P \vee A$;
- $(P \vee \perp)^{rF} \equiv P \vee A$.

We conclude that the realisability method with K versus K_1 is the same.

4 Second variant of the Kuroda negative translation

4.1 Motivation

In section 3.1 we presented the idea of composing K with a translation T_1 from IL into ML , getting $K_1 = T_1 \circ K$. Now, following the same idea but using a different translation T_2 of IL into ML , we present a new composition $K_2 = T_2 \circ K$. The new translation T_2 simply double negates all atomic formulas different from \perp .

$$\begin{array}{ccccc} CL & \xrightarrow{K} & IL & \xrightarrow{T_2} & ML \\ & \searrow & & \nearrow & \\ & & K_2 & & \end{array}$$

4.2 Translation T_2

Definition 11 ([21, page 686]). The translation T_2 maps each formula A to the formula A^{T_2} obtained from A by simultaneously substituting in A all atomic subformulas $P \neq \perp$ by $\neg\neg P$.

Theorem 12 (soundness of T_2 [21, page 686]). *For all formulas A , we have: if $IL \vdash A$, then $ML \vdash A^{T_2}$.*

Proof. Analogous to the alternative proof of the soundness theorem of T_1 . □

Theorem 13 (characterisation of T_2). *For all formulas A , we have $CL \vdash A \leftrightarrow A^{T_2}$.*

Proof. Just note that A^{T_2} is obtained from A by substituting in A the atomic subformulas $P \neq \perp$ by the equivalent (in CL) $\neg\neg P$. □

4.3 Translation K_2

Definition 14 ([2, page 21]). The *second variant* K_2 of the Kuroda negative translation maps each formula A to the formula $A^{K_2} := \neg\neg A_{K_2}$, where A_{K_2} is defined by induction on the structure of formulas by

$$\begin{aligned} P_{K_2} &:= \neg\neg P, & (A \wedge B)_{K_2} &:= A_{K_2} \wedge B_{K_2}, \\ \perp_{K_2} &:= \perp, & (A \vee B)_{K_2} &:= A_{K_2} \vee B_{K_2}, \\ (\forall x A)_{K_2} &:= \forall x \neg\neg A_{K_2}, & (A \rightarrow B)_{K_2} &:= A_{K_2} \rightarrow B_{K_2}, \\ (\exists x A)_{K_2} &:= \exists x A_{K_2}, \end{aligned}$$

where $P \not\equiv \perp$ is an atomic formula.

Remark 15. The only difference between K and K_2 is in the clause of P .

Proposition 16 (factorisation $K_2 = T_2 \circ K$). *For all formulas A , we have $A^{K_2} \equiv (A^K)^{T_2}$.*

Proof. Analogously to the proof of the factorisation $K_1 = T_1 \circ K$. \square

Theorem 17 (soundness of K_2). *For all formulas A , we have: if $CL \vdash A$, then $ML \vdash A^{K_2}$.*

Proof. Analogous to the proof of the soundness theorem of K_1 . \square

Theorem 18 (characterisation of K_2). *For all formulas A , we have $CL \vdash A \leftrightarrow A^{K_2}$.*

Proof. Analogous to the proof of the characterisation theorem of K_1 . \square

Theorem 19 (equivalence between K and K_2).

1. *For all formulas A , we have $IL \vdash A^K \leftrightarrow A^{K_2}$.*
2. *The previous point does not hold in ML .*

Proof.

1. We do the proof by induction on the structure of formulas. Let us consider only the cases of atomic formulas $P \not\equiv \perp$, \vee and \exists . The remaining cases are analogous. In the following, “IH” marks a use of induction hypothesis. It is convenient to recall that IL proves $\neg\neg C \leftrightarrow \neg\neg\neg\neg C$, $\neg\neg(C \vee D) \leftrightarrow \neg\neg(\neg\neg C \vee \neg\neg D)$ and $\neg\neg\exists x C \leftrightarrow \neg\neg\exists x \neg\neg C$. In IL we have:

$$\begin{aligned} & (A \vee B)^K \equiv & (\exists x A)^K \equiv \\ & \neg\neg(A_K \vee B_K) \leftrightarrow & \neg\neg\exists x A_K \leftrightarrow \\ P^K \equiv & \neg\neg(\neg\neg A_K \vee \neg\neg B_K) \equiv & \neg\neg\exists x \neg\neg A_K \equiv \\ \neg\neg P \leftrightarrow & \neg\neg(A^K \vee B^K) \stackrel{IH}{\leftrightarrow} & \neg\neg\exists x A^K \stackrel{IH}{\leftrightarrow} \\ \neg\neg\neg\neg P \equiv & \neg\neg(A^{K_2} \vee B^{K_2}) \equiv & \neg\neg\exists x A^{K_2} \equiv \\ P^{K_2}, & \neg\neg(\neg\neg A_{K_2} \vee \neg\neg B_{K_2}) \leftrightarrow & \neg\neg\exists x \neg\neg A_{K_2} \leftrightarrow \\ & \neg\neg(A_{K_2} \vee B_{K_2}) \equiv & \neg\neg\exists x A_{K_2} \equiv \\ & (A \vee B)^{K_2}, & (\exists x A)^{K_2}. \end{aligned}$$

2. Analogous to the proof of the second point of the equivalence between K and K_1 . \square

4.4 Effect on extracting computational content

Let us recall that the only difference between K and K_2 is $P_K \equiv P$ and $P_{K_2} \equiv \neg\neg P$ (where $P \not\equiv \perp$ is an atomic formula).

Functional interpretation method To compare the functional interpretation method with K versus K_2 , we only have to compare P^D versus $(\neg\neg P)^D$:

- $P^D \equiv P$;
- $(\neg\neg P)^D \equiv \neg\neg P$, and in Heyting arithmetic (where atomic formulas are decidable) $\neg\neg P$ is equivalent to P .

We conclude that the functional interpretation method with K versus K_2 is (essentially) the same.

Realisability method To compare the realisability method with K versus K_1 , we only have to compare P^F versus $(\neg\neg P)^{rF}$:

- $P^F \equiv P \vee A$;
- $(\neg\neg P)^{rF} \equiv (P \rightarrow A) \rightarrow A$, and in Heyting arithmetic $(P \rightarrow A) \rightarrow A$ is equivalent to $P \vee A$.

We conclude that the realisability method with K versus K_2 is (essentially) the same.

5 Third variant of the Kuroda negative translation

5.1 Motivation

The Kuroda negative translation K does not translate **CL**, based on $\perp, \wedge, \vee, \rightarrow, \forall$ and \exists , into **ML**, but (as remarked by Benno van den Berg) does translate **CL**, based on $\perp, \neg, \wedge, \vee, \forall$ and \exists , into **ML**. The main difference between the two bases is that in the latter $B \rightarrow C$ stands for $\neg B \vee C$. This suggests us that instead of directly translating a formula via K , we should:

- first translate A by a translation T_3 that substitutes all subformulas of the form $B \rightarrow C$ by $\neg B \vee C$;
- and then translate via K .

Doing so, we get a translation $K_3 = K \circ T_3$ of **CL** into **ML**.

$$\begin{array}{ccccc} \text{CL} & \xrightarrow{T_3} & \text{CL} & \xrightarrow{K} & \text{ML} \\ & & \searrow & \nearrow & \\ & & & K_3 & \end{array}$$

5.2 Translation T_3

Definition 20. The translation T_3 maps each formula A to the formula A^{T_3} obtained from A by simultaneously substituting in A all subformulas of the form $B \rightarrow C$ by $\neg B \vee C$.

Theorem 21 (soundness of T_3). *For all formulas A , we have: if $CL \vdash A$, then $CL \vdash A^{T_3}$.*

Proof. Follows from the characterisation theorem of T_3 below. \square

Theorem 22 (characterisation of T_3). *For all formulas A , we have $CL \vdash A \leftrightarrow A^{T_3}$.*

Proof. Just note that T_3 replaces $B \rightarrow C$ by the equivalent (in CL) $\neg B \vee C$. \square

5.3 Translation K_3

Definition 23 ([6, section 1]). The *third variant K_3 of the Kuroda negative translation* maps each formula A to the formula $A^{K_3} := \neg\neg A_{K_3}$, where A_{K_3} is defined by induction on the structure of formulas by

$$\begin{aligned} P_{K_3} &:= P, & (A \wedge B)_{K_3} &:= A_{K_3} \wedge B_{K_3}, \\ (\forall x A)_{K_3} &:= \forall x \neg\neg A_{K_3}, & (A \vee B)_{K_3} &:= A_{K_3} \vee B_{K_3}, \\ (\exists x A)_{K_3} &:= \exists x A_{K_3}, & (A \rightarrow B)_{K_3} &:= \neg A_{K_3} \vee B_{K_3}, \end{aligned}$$

where P is an atomic formula.

Remark 24. The only difference between K and K_3 is in the clause of \rightarrow .

Remark 25. Ilik [16, definition 3.1], and Ferreira and Oliva [7, section 6.3], presented a variant K'_3 of the Kuroda negative translation defined like K_3 except for $(A \rightarrow B)_{K'_3} := A_{K'_3} \rightarrow \neg\neg B_{K'_3}$. This translation K'_3 is minimally equivalent to K_3 , that is for all formulas A we have $ML \vdash A^{K_3} \leftrightarrow A^{K'_3}$.

Proposition 26 (factorisation $K_3 = K \circ T_3$). *For all formulas A , we have $A^{K_3} \equiv (A^{T_3})_K$.*

Proof.

1. First, we prove $(A^{T_3})_K \equiv A_{K_3}$ by induction on the structure of formulas. Let us consider only the cases of atomic formulas P , \wedge , \rightarrow and \forall . The remaining cases are analogous. In the following, “IH” marks a use of induction hypothesis.

$$\begin{array}{llll} (P^{T_3})_K \equiv & ((A \wedge B)^{T_3})_K \equiv & ((A \rightarrow B)^{T_3})_K \equiv & ((\forall x A)^{T_3})_K \equiv \\ & (A^{T_3} \wedge B^{T_3})_K \equiv & (\neg A^{T_3} \vee B^{T_3})_K \equiv & (\forall x A^{T_3})_K \equiv \\ P_K \equiv & & & \\ P \equiv & (A^{T_3})_K \wedge (B^{T_3})_K \stackrel{\text{IH}}{=} & \neg(A^{T_3})_K \vee (B^{T_3})_K \stackrel{\text{IH}}{=} & \forall x \neg\neg(A^{T_3})_K \stackrel{\text{IH}}{=} \\ P_{K_3}, & A_{K_3} \wedge B_{K_3} \equiv & \neg A_{K_3} \vee B_{K_3} \equiv & \forall x \neg\neg A_{K_3} \equiv \\ & (A \wedge B)_{K_3}, & (A \rightarrow B)_{K_3}, & (\forall x A)_{K_3}. \end{array}$$

2. Now we prove $(A^{T_3})^K \equiv A^{K_3}$ using $(A^{T_3})_K \equiv A_{K_3}$:

$$(A^{T_3})^K \equiv \neg\neg(A^{T_3})_K \equiv \neg\neg A_{K_3} \equiv A^{K_3}. \quad \square$$

Theorem 27 (soundness of K_3 [6, section 1]). *For all formulas A , we have: if $CL \vdash A$, then $ML \vdash A^{K_3}$.*

Proof.

1. Having in mind the factorisation $K_3 = K \circ T_3$, just compose the soundness theorems of K (as a translation from CL based on $\perp, \neg, \wedge, \vee, \forall$ and \exists into ML) and T_3 .
2. We did not find in the literature a proof of the soundness of K for CL based on $\perp, \neg, \wedge, \vee, \forall$ and \exists , so we sketch a proof. We extend the definition of K to the primitive negation \neg by $(\neg A)_K := \neg A_K$. We consider the axioms and rules of CL based on \neg, \vee and \forall (where $A \rightarrow B := \neg A \vee B$ and $A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A)$) given by Shoenfield [23, sections 2.6 and 8.3] and listed on the left column from (1) to (7). To get a system for CL based on $\perp, \neg, \wedge, \vee, \forall$ and \exists , we add to Shoenfield's axioms and rules the axioms on the left column from (8) to (10) (the axiom $\perp \rightarrow A \equiv \neg\perp \vee A$ is seemly missing, but is actually derivable from (10)). On the right column we write the translation by K of the axioms and rules.

$$\neg A \vee A \qquad \neg\neg(\neg A_K \vee A_K) \quad (1)$$

$$\forall x A \rightarrow A[t/x] \qquad \neg\neg(\neg\forall x\neg\neg A_K \vee A_K[t/x]) \quad (2)$$

$$\frac{A}{B \vee A} \qquad \frac{\neg\neg A_K}{\neg\neg(B_K \vee A_K)} \quad (3)$$

$$\frac{A \vee A}{A} \qquad \frac{\neg\neg(A_K \vee A_K)}{\neg\neg A_K} \quad (4)$$

$$\frac{A \vee (B \vee C)}{(A \vee B) \vee C} \qquad \frac{\neg\neg(A_K \vee (B_K \vee C_K))}{\neg\neg((A_K \vee B_K) \vee C_K)} \quad (5)$$

$$\frac{A \vee B \quad \neg A \vee C}{B \vee C} \qquad \frac{\neg\neg(A_K \vee B_K) \quad \neg\neg(\neg A_K \vee C_K)}{\neg\neg(B_K \vee C_K)} \quad (6)$$

$$\frac{A \vee B}{\forall x A \vee B} \qquad \frac{\neg\neg(A_K \vee B_K)}{\neg\neg(\forall x\neg\neg A_K \vee B_K)} \quad (7)$$

$$(A \wedge B) \leftrightarrow \neg(\neg A \vee \neg B) \qquad \neg\neg((\neg(A_K \wedge B_K) \vee \neg(\neg A_K \vee \neg B_K)) \wedge (\neg\neg(\neg A_K \vee \neg B_K) \vee (A_K \wedge B_K))) \quad (8)$$

$$\exists x A \leftrightarrow \neg\forall x\neg A \qquad \neg\neg((\neg\exists x A_K \vee \neg\forall x\neg\neg A_K) \wedge (\neg\neg\forall x\neg\neg A_K \vee \exists x A_K)) \quad (9)$$

$$\neg\perp \qquad \neg\neg\neg\perp \quad (10)$$

Then there is the tedious task of verifying that the formulas and rules in the right column hold in ML . Let us, for example, verify (7). Using $ML \vdash$

$\neg(C \vee D) \leftrightarrow \neg C \wedge \neg D$ we see that (7) is equivalent to $\frac{\neg(\neg A_K \wedge \neg B_K)}{\neg(\neg \forall x \neg \neg A_K \wedge \neg B_K)}$. Let us assume $\neg(\neg A_K \wedge \neg B_K)$ and $\neg \forall x \neg \neg A_K \wedge \neg B_K$, and conclude \perp . If $\neg A_K$, then $\neg A_K \wedge \neg B_K$, contradicting the first assumption, so $\neg \neg A_K$. Then $\forall x \neg \neg A_K$, contradicting the second assumption, thus we conclude \perp , as we wanted. \square

Theorem 28 (characterisation of K_3). *For all formulas A , we have $\mathbf{CL} \vdash A \leftrightarrow A^{K_3}$.*

Proof. Analogous to the proof of the characterisation theorem of K_1 . \square

Theorem 29 (equivalence between K and K_3).

1. *For all formulas A , we have $\mathbf{IL} \vdash A^K \leftrightarrow A^{K_3}$.*
2. *The previous point does not hold in \mathbf{ML} .*

Proof.

1. By the characterisation theorem of T_3 we have $\mathbf{CL} \vdash A \leftrightarrow A^{T_3}$. Then by the soundness theorem of K we get $\mathbf{IL} \vdash (A \leftrightarrow A^{T_3})^K$. But $\mathbf{IL} \vdash (A \leftrightarrow A^{T_3})^K \leftrightarrow (A^K \leftrightarrow (A^{T_3})^K)$ as we argue now (using $\mathbf{IL} \vdash \neg \neg(B \leftrightarrow C) \leftrightarrow (\neg \neg B \leftrightarrow \neg \neg C)$):

$$\begin{aligned} (A \leftrightarrow A^{T_3})^K &\equiv \\ \neg \neg (A \leftrightarrow A^{T_3})_K &\equiv \\ \neg \neg (A_K \leftrightarrow (A^{T_3})_K) &\leftrightarrow \\ (\neg \neg A_K \leftrightarrow \neg \neg (A^{T_3})_K) &\equiv \\ A^K \leftrightarrow (A^{T_3})^K. \end{aligned}$$

So $\mathbf{IL} \vdash A^K \leftrightarrow (A^{T_3})^K$ where $A^{K_3} \equiv (A^{T_3})^K$ by the factorisation $K_3 = K \circ T_3$.

2. Analogous to the proof of the second point of the equivalence between K and K_1 . \square

5.4 Effect on extracting computational content

Let us recall that the only difference between K and K_3 is $(A \rightarrow B)_K \equiv A_K \rightarrow B_K$ and $(A \vee B)_{K_3} \equiv \neg A_{K_3} \vee B_{K_3}$.

Functional interpretation method To compare the functional interpretation method with K versus K_3 , we only have to compare $(A_K \rightarrow B_K)^D$ versus $(\neg A_{K_3} \vee B_{K_3})^D$:

- $(A_K \rightarrow B_K)^D \equiv \exists \underline{U}, \underline{Y} \forall \underline{x}, \underline{v} ((A_K)_D(\underline{x}; \underline{Y} \underline{x} \underline{v}) \rightarrow (B_K)_D(\underline{U} \underline{x}; \underline{v}))$;
- $(\neg A_{K_3} \vee B_{K_3})^D \equiv \exists n, \underline{Y}, \underline{u} \forall \underline{x}, \underline{v} ((n = 0 \rightarrow \neg (A_{K_3})_D(\underline{x}; \underline{Y} \underline{x})) \wedge (n \neq 0 \rightarrow (B_{K_3})_D(\underline{u}; \underline{v})))$, where

- on the positive side the variables \underline{Y} have a lower type than the corresponding ones for $(A_K \rightarrow B_K)^D$,
- on the negative side it is introduced one more variable n .

Realisability method To compare the realisability method with K versus K_3 , we count the number of occurrences of A introduced by F and rF in $(B^K)^F$ and $(B^{K_3})^{rF}$. Say that B has n_P atomic subformulas $P \not\equiv \perp$, n_\perp occurrences of \perp , n_\rightarrow implications \rightarrow , and n_\forall quantifiers \forall . Note that in a negation $\neg C \equiv C \rightarrow \perp$ there is an occurrence of \perp .

- The translation F introduces in $(B^K)^F$ one occurrence of A for each atomic subformula of B^K . So we have to count the number of atomic subformulas of B^K :

$$\begin{array}{rcl}
n_P & & \text{(one for each } P \not\equiv \perp \text{ inherited from } B) \\
n_\perp & & \text{(one for each } \perp \text{ inherited from } B) \\
2n_\forall & & \text{(two for each } \neg\neg \text{ introduced by } (\forall x C)_K \equiv \forall x \neg\neg C_K) \\
+ & 2 & \text{(two for the } \neg\neg \text{ introduced by } B^K \equiv \neg\neg B_K) \\
\hline
n_P + n_\perp + 2n_\forall + 2 & &
\end{array}$$

- The translation rF introduces in $(B^{K_3})^{rF}$ one occurrence of A for each occurrence of \perp in B^{K_3} . So we have to count the number of occurrences of \perp in B^{K_3} :

$$\begin{array}{rcl}
n_\rightarrow & & \text{(one for each } \neg \text{ introduced by } (C \rightarrow D)_{K_3} \equiv \neg C_{K_3} \vee D_{K_3}) \\
n_\perp & & \text{(one for each } \perp \text{ inherited from } B) \\
2n_\forall & & \text{(two for each } \neg\neg \text{ introduced by } (\forall x C)_{K_3} \equiv \forall x \neg\neg C_{K_3}) \\
+ & 2 & \text{(two for the } \neg\neg \text{ introduced by } B^{K_3} \equiv \neg\neg B_{K_3}) \\
\hline
n_\rightarrow + n_\perp + 2n_\forall + 2 & &
\end{array}$$

The number of occurrences of A in $(B^K)^F$ is strictly smaller than the number of occurrences of A in $(B^{K_3})^F$ if and only if $n_P < n_\rightarrow$. We conclude that the realisability method with K has fewer occurrences of A than the realisability method with K_3 if and only if $n_P < n_\rightarrow$. (Actually, for a complete account of the complexity of the realisability method with K versus K_3 , we should still considered that a realisability is applied to $(B^K)^F$ and $(B^{K_3})^{rF}$, but to keep the comparison simple we only look at the number of occurrences of A .)

6 Conclusion

The Kuroda negative translation K translates CL into IL, but not into ML. We presented three variants K_1 , K_2 and K_3 of K that translate CL into ML. Also, we discussed how they affect the methods of extracting computational content from proofs. We summarise in the following table the comparison between the variants and K.

Variant	Difference	Effect on extraction of computational content
K_1	$P_{K_1} := P \vee \perp$	equal
K_2	$P_{K_2} := \neg\neg P$	equal
K_3	$(A \rightarrow B)_{K_3} := \neg A_{K_3} \vee B_{K_3}$	lower types, more variables, more/fewer occurrences of A

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